Minimization of numerical dispersion errors in 2D finite element models of short acoustic wave propagation

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Abstract. Numerical dispersion errors are inherent for simulations based on wave propagation models with discrete meshes. The paper presents approach applied to reduce errors of this type in finite element models. High order 2D synthesized finite elements with enhanced convergence properties are obtained by modal synthesis technique. Obtained elements have diagonal mass matrix which enables to employ explicit integration schemes for wave simulation. Waves of more than two times wider frequency can be simulated using model of synthesized elements compared to models assembled of conventional elements. Furthermore, such elements could be used as a template of higher-order element to construct finite element models for all simulation problems of this kind.

Keywords: finite elements, wave propagation, numerical dispersion, modal synthesis.

1 Introduction

The work refers to short-wave simulation finite element (FE) models where the range of investigated wavelengths many times shorter than the characteristic length of the propagation environment. As an example, acoustic waves in solids, hydraulic pressure pulses propagation in large pipeline networks, etc. can be presented as short-wave propagation models. They require huge amount of elements (even "small" models for 2D problems contain $10^6 - 10^7$ elements) and therefore computational resources required for simulation are very large. Generally, the dimensionality of the FE model is reduced as rougher meshes are applied, where measure for roughness of the mesh is the number of elements per characteristic wavelength. Unfortunately, rough meshes tend to increase the simulation errors, which exhibit themselves as severe deterioration of the shapes of propagating wave pulses as the time of simulation increases. In other words, they can be referred to as numerical dispersion and phase velocity errors.

Already in 1980s researchers have noticed different modal convergence features of dynamic models obtained by using lumped and consistent forms of

adfa, p. 1, 2016.

mass matrices [1]. The simplest way to reduce the numerical dispersion of dynamic models is to use the 'combined' form of the mass matrix obtained as a weighted superposition of the two traditional forms [2]. However, the models with non-diagonal mass matrix were unable to fully exploit the advantages of explicit time integration schemes. In recent years this problem has been examined more thoroughly. Generally, the results are obtained by using models based on the higher order FE, which could ensure the sufficient accuracy of simulation results within acceptable limits. In [3] equidistant, Lobatto and Chebyshev nodal positions within a FE were investigated. The positioning of nodes has been demonstrated to be important in case of higher-order FE. In [4] the general template for retrieving characteristic matrices of n-node bar elements based on their reduced diagonal representations has been proposed. In [5] two different formulations based on the modified integration rule for the mass and stiffness matrices and on the averaged mass matrix has been introduced. These techniques with reduced dispersion for linear elastodynamics problems which enable reduce the numerical dispersion for linear FE models has been investigated.

The approach presented in this work is based on synthesized finite elements (SFE). Synthesis of elements is performed by minimizing the penalty-type target function, where the design variables are modal shapes and mode frequencies of elements. The target function evaluates the magnitude of the phase velocity error in terms of the modal frequencies of the sample model consisting of SFE. Originally the method was proposed in [6] for 1D case. In [7] it has been expanded to 2D case. There models consisting of SFE preserve small phase velocity errors in meshes that are 3-5 times rougher than the ones required for conventional lumped mass matrix models. The novelty of this work is that only stiffness matrix of the model is computed by modal synthesis, while the diagonal mass matrix remains unchanged. This enables to use explicit integration schemes for wave simulation. The numerical results obtained during 2D acoustic wave pulse simulation are analyzed.

2 Synthesis of the finite element

The finite element model of elastic media in which the propagating acoustic wave is considered is derived from general structural dynamic equation system

$$[M]\{\ddot{U}\} + [K]\{U\} = \{F(t)\} \tag{1}$$

where $[M]$ and $[K]$ are mass and stiffness matrices, $\{U\}$ is the nodal displacement vector and $\{F(t)\}$ is the excitation force vector. Modal frequencies (MF) and modal shapes (MS) of the structure are obtained by solving the eigenvalue problem as

$$([K] - \omega^2[M])\{y\} = \{0\} \quad (2)$$

where ω is modal angular frequency and $\{y\}$ is the modal shape.

For real and symmetric structural matrices $[M]$ and $[K]$ the solutions of (2) equation are obtained as structural modes $\omega_i, \{y_i\}, i = 1, \dots, n$, where n – is degree of freedom of the structure. The fundamental properties of structural modes provide derivation of [K] matrix in terms of normalized MS and MF as

$$[K] = ([Y]^T)^{-1}[diag(\omega_1{}^2, \omega_2{}^2, \dots, \omega_n{}^2)][Y]^{-1} \quad (3)$$

where $[Y] = [\{y_1\}, \{y_2\}, \dots, \{y_n\}]$ is the matrix of MS.

Relationship (3) means that the stiffness matrix of an element, of a substructure or of a structure can be generated by directly referring to their known or desired values of MF and MS. The idea of our approach is to find such modes used for synthesizing an element that the cumulative modal frequency error of the sample domain (SD) assembled of synthesized elements would be as small as possible. The outline of the synthesis procedure is presented in Fig 1.

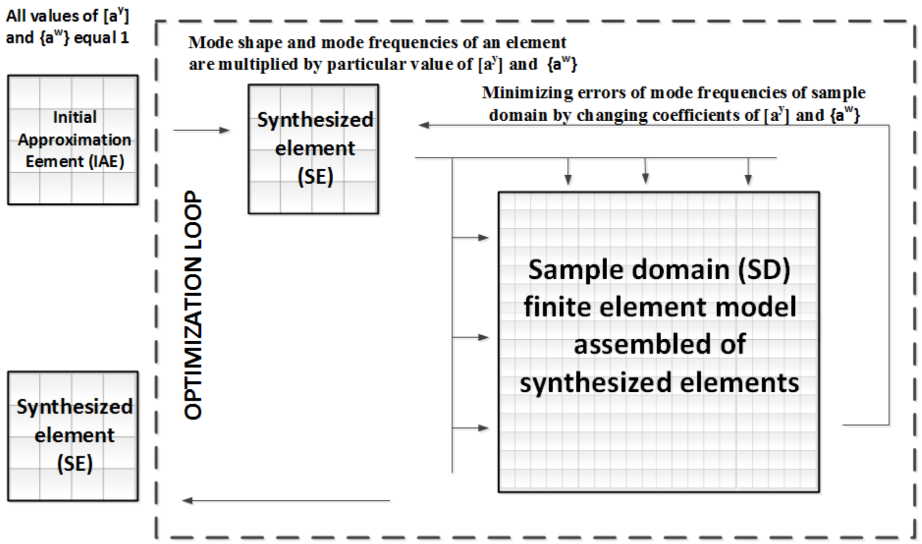

Fig. 1. Outline of the synthesis procedure

In this work structure assembled from conventional square-form elements with lumped mass matrix is used as initial approximation element (IAE). During the optimization loop, MS and MF of synthesized element (SE) were slightly modified in order to ensure that the sample domain assembled of a certain number of synthesized elements provides as many as possible close-to-exact modes.

The optimization loop in Fig. 1 is used for the minimization of the target function Ψ, which presents the cumulative error of modal frequencies of the SD as

$$\min_{[a^y],\{a^\omega\}} \Psi = \sum_{i=1}^{\tilde{N}} \left(\frac{\tilde{\omega}_i - \omega_{i0}}{\omega_{i0}}\right)^2 \tag{4}$$

where $\{\tilde{\omega}_0, \dots, \tilde{\omega}_{\tilde{N}}\}$ are the MFs of the SD assembled of SE, $\{\hat{\omega}_{00}, \dots, \hat{\omega}_{0\tilde{N}}\}$ are close-to-exact MFs of the SD, and $\{a^\omega\}$, $[a^y]$ are the vector and matrix of MF and MS correction coefficients correspondingly, used as optimization variables. The summation of errors is performed over $\tilde{N} \leq N$ modal frequencies of the SD, where N is the number of modes of the SD. The close-to exact modal frequencies have to be computed only once, by using densely meshed FE model or in certain cases they are known analytically.
The correction of modal shapes and modal frequencies is performed as

$$[\tilde{Y}] = \left[\{y_{11} * a_{11}^y, \dots, y_{1n} * a_{1n}^y\}, \dots, \{y_{n1} * a_{n1}^y, \dots, y_{nn} * a_{nn}^y\}\right] \tag{5}$$

$$\left[\mathrm{diag}\left(\tilde{\omega}_1{}^2, \dots, \tilde{\omega}_n{}^2\right)\right] = [\mathrm{diag}(\omega_1 * a_1^\omega, \dots, \omega_n * a_n^\omega)] \tag{6}$$

Each j-th term of i-th MS and i-th MF is multiplied by the corresponding value taken from matrix $[a^y]$ and vector $\{a^\omega\}$. Corrections are performed for all MSs with exception of the rigid-body modal shapes, which correspond to zero modal frequencies. However, for pairs of MS having the same MF only one MS is corrected, while the other is derived from the first corrected MS. One may think that the result is dependent on the selected size and shape of the SD, which is selected freely. However our numerical experiments demonstrated that even the usage of SD of modest size enables to obtain good results, as described in the next subsection. Generally, higher order of the SE and large dimensionality of SD enables to construct a better synthesized element. However, this leads to drastic increase of optimization variables. Practically, a compromise has been sought between the necessary computational resource and the quality of the synthesized element.

3 Minimization of modal frequency errors

As a numerical example, the analysis of wave propagation in a 2D acoustic environment has been performed. A quadratic SE of 5x5 nodes has been constructed. Matrices of the first order conventional FE with lumped mass matrix read as

$$[M^e] = \frac{\rho * S^e}{4}[I] \tag{7}$$

$$[K^e] = E * S^e [B]^T [B] \tag{8}$$

where E – Young's modulus, ρ – mass density, S^e – area of the element, $[I]$ and $[B]$ are identity and strain matrices. Stiffness matrix $[K^{IA}]$ of the initial approximation element (IAE), computed by using (3) with modes obtained by solving eigenvalue problem (2) for structure assembled of 16×16 conventional 2x2 nodes finite elements. Mass matrix $[M^{IA}]$ of IAE remains diagonal and is the same as the structural mass matrix assembled of conventional elements. Synthesized element stiffness matrix reads as

$$[K^{SE0}] = \left([\tilde{Y}]^T\right)^{-1}\left[\mathrm{diag}\left({\tilde{\omega}_1}^2, {\tilde{\omega}_2}^2, \ldots, {\tilde{\omega}_n}^2\right)\right][\tilde{Y}]^{-1} \tag{9}$$

where $[\tilde{Y}]$ and $\left[\mathrm{diag}\left({\tilde{\omega}_1}^2, {\tilde{\omega}_2}^2, \ldots, {\tilde{\omega}_n}^2\right)\right]$ are the MS and MF obtained after element synthesis, while constants ρ, E, S^e have been assigned the value 1 for the synthesis. The target function included the first 25% of MF of the sample domain ($\tilde{N} = 0.25 * N$). Close-to-exact MFs of SD used as reference values in the target function have been obtained by solving eigenvalue problem (2) for the structure meshed with 5 times smaller linear dimension of FE. Results of the synthesis in terms of relative modal frequency errors as $\frac{\widehat{\omega}_i - \omega_{i0}}{\omega_{i0}}$ are shown in Fig. 2.

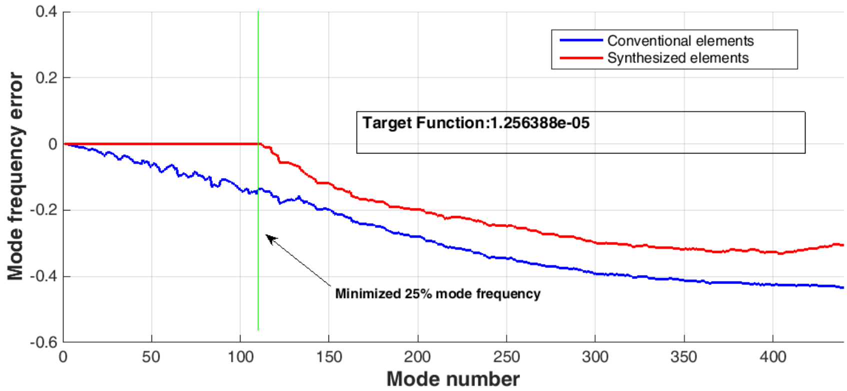

Fig. 2. Modal frequency errors of the sample domain assembled of 25 synthesized elements with minimized cumulative error of 25% of modal frequencies

From the results of Fig. 2 it can be seen that the synthesis process worked well. First 25% of MF of the model assembled of synthesized elements was very close-to-exact as was required. The final value of the target function value was $\sum_{i=1}^{\tilde{N}} \left(\frac{\widehat{\omega}_i - \omega_{i0}}{\omega_{i0}}\right)^2 \approx 1.2^{-5}$. Elements obtained by the synthesis procedure could be used as a template

$$[K^{SE}] = E * [K^{SE0}] \tag{10}$$

for constructing finite element stiffness matrices for all simulation problems of this kind as a higher-order element stiffness matrix. Mass matrix of the model remains diagonal and is the same as of the structure assembled of conventional elements.

4 Numerical investigation with application to 2D acoustic wave propagation

The numerical experiments have been carried out by investigating the wave propagation in water, where 2D rectangular structure model of 0.32 m x 0.24 m was assembled of conventional and synthesized elements using *0.5 mm* finite element mesh size (**Fig. 3a**). Physical constants were $\rho = 995\ \mathrm{kg/m^3}$ and $E = 2.2\ \mathrm{GPa}$, the phase velocity of the wave is $v = \sqrt{E/\rho} = 1487\ \mathrm{m/s}$. The excitation pulse was a sine wave multiplied by a Gaussian window:

$$u(t) = e^{-a(t-b)^2} sin(2\pi f t) \tag{11}$$

where $a = k_a f \sqrt{-\frac{2\ln 0.1}{p_s}}$; $b = 2p_s/3f$, p_s is the number of periods, k_a is the asymmetry factor and f is the frequency [8]. Simulation was carried by simulating short period pulse as $p_s = 1.5$ at excitation zone (**Fig. 3b**).

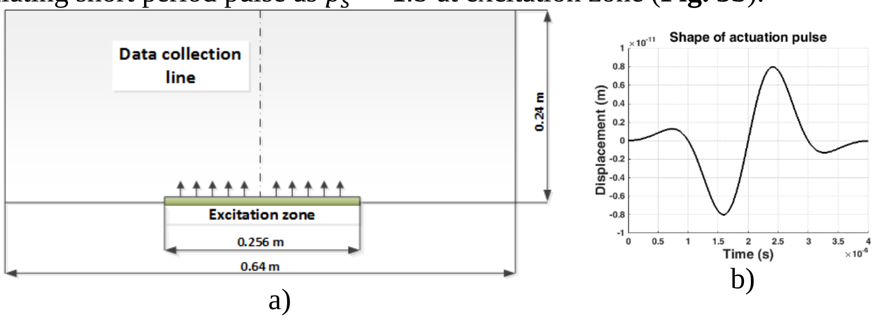

Fig. 3. a) Geometry of the finite element model b) Excitation pulse shape

Pulse has been actuated for 4 μs at excitation zone and its propagation for 240 μs has been simulated. Fig. 4 shows the simulated B-Scan image, where black pattern refers to the amplitude of the pulse along the data collection line at different time moments.

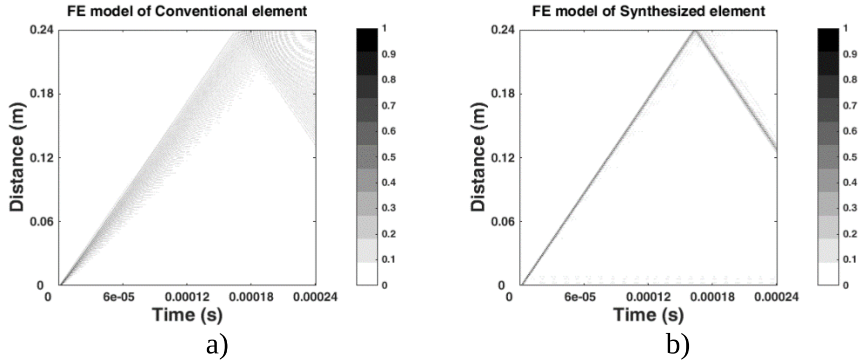

Fig. 4. B-scans of pulse propagation in model assembled of a) conventional element; b) synthesized elements

Analysis of the simulation results in Fig. 4 shows that when the pulse is simulated using the model of conventional elements, the numerical distortions grow with time and at the end of simulation the pulse is highly distorted, while in model of SE distortion remains small. Dispersion analysis practically cannot be performed by solving eigenvalue problem (2), because mass and stiffness matrices of the model have size of 616161 x 616161. In order to estimate the character of the observed dispersion the B-scan data have been converted from space-time domain into phase velocity - frequency domain by using 2D Fourier transform. Obtained images of dispersion curves are presented in Fig.5 where phase velocity in the images at different frequencies corresponds to the yellow pattern.

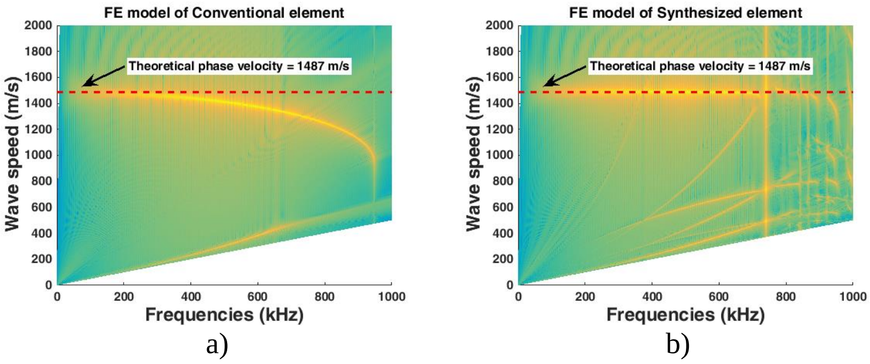

Fig. 5. Dispersion curve of the FE model assembled of a) conventional elements; b) synthesized elements

The comparison of dispersion curves of the models assembled of SE and CE leads to the conclusion that in model of CE phase velocity is close to theoretical till ~ 250 kHz and at higher frequencies phase velocity inaccuracies grow rather quickly (Fig. 5a). On the contrary, the model assembled of SE retains a good accuracy of phase velocity till ~ 700 kHz (Fig. 5b). This means that models assembled of synthesized elements can be used to simulate the propagation of wave pulses of more than two times wider frequency than the models assembled of conventional elements at the same mesh density.

5 Conclusions

The approach for the reduction of the numerical dispersion in two-dimensional model has been proposed. Fifth order synthesized finite element obtained using modal synthesis technique, where first quarter of mode frequencies of models assembled of synthesized element are close to exact. 2D finite element model of water material has been assembled of conventional and synthesized elements. By investigating driving ultrasonic pulse, numerical dispersion has been compared for models assembled of synthesized and conventional elements. Results show that in models of synthesized elements numerical dispersion is close to zero over more than two times wider frequency range compared against the models assembled of conventional elements at the same mesh density.

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